BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 5

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CONTENTS

I. REVIEW

Last time we:

(1) Stated the maximum modulus principle and used it to prove Schwarz's lemma.
(2) Computed $\mathrm{Aut}(\mathfrak{D})$ and $\mathrm{Aut}(\mathfrak{H})$ using Schwarz's lemma.
(3) Defined the order of vanishing of a meromorphic function at a point, and defined the ramification index of a morphism at a point.

II. MORE LOCAL PROPERTIES OF MORPHISMS

II.1. Order of vanishing and ramification.

Definition 1. Let X be a Riemann surface, $P \in X$, and $f \in \mathcal{M}(X)$ be a meromorphic function. Let φ be a centered coordinate map at P, so $\varphi(P) = 0$. Then f can represented by the Laurent series $f \circ \varphi^{-1}(z) = \sum_n a_n z^n$. The order (of vanishing) of f at P, denoted by $\mathrm{ord}_P(f)$ is the smallest n such that $a_n \neq 0$:

$$\mathrm{ord}_P(f) := \min\{n \in \mathbb{Z} : a_n \neq 0\}.$$

If $\mathrm{ord}_P(f) n \geq 1$, then f has a zero of order n at P and if $\mathrm{ord}_P(f) = -n < 0$, then f has a pole of order n at P.

Definition 2. Let $f : X \to Y$ be a morphism of Riemann surfaces, $P \in X$. Let ψ be a chart of Y centered at $f(P)$, so $\psi(f(P)) = 0$. The integer $e_P(f)$ or $m_P(f)$ given by

$$e_P(f) := \mathrm{ord}_P(\psi \circ f)$$

is the ramification index or multiplicity of f at P. Equivalently,

$$e_P(f) = 1 + \mathrm{ord}_P(\psi \circ f)'$$

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whether ψ is a centered chart or not.

If $e_P(f) \geq 2$, then $P \in X$ is ramification point or branch point of f, with ramification index $e_P(f)$. A branch value is the image of a ramification point. Equivalently, we say that f is ramified above $Q \in Y$ if there is some $P \in f^{-1}(Q)$ with $e_P(f) \geq 2$ and f is ramified at $P \in X$ if $P \in X$ and $e_P(f) \geq 2$.

By choosing our charts judiciously, we can actually find a local representation of a morphism of the form $z \mapsto z^m$.

Proposition 3 (Local Normal Form). *Let $F : X \to Y$ be a nonconstant morphism of Riemann surfaces. Fix $P \in X$ and let $m = e_P(F)$. Then for every chart $\psi : V \to \widehat{V}$ on Y centered at $F(P)$, there exists a chart $\varphi : U \to \widehat{U}$ on X centered at P such that*

$$(\psi \circ F \circ \varphi^{-1})(z) = z^m .$$

Proof. Fix a chart ψ on Y centered at $F(P)$ (i.e., $\psi(F(P)) = 0$), and choose any chart $\theta : W \to \widehat{W}$ centered at P. Then the Taylor series for the function $T(w) := (\psi \circ F \circ \theta^{-1})(w)$ is of the form

$$T(w) = \sum_{j=m}^{\infty} c_j w^j$$

where $c_m \neq 0$ and $m = m_P(F)$. (Since we picked a centered chart, we have $T(0) = 0$.) Factoring out w^m, we have $T(w) = w^m S(w)$ where S is a holomorphic function at $w = 0$ and $S(0) \neq 0$. Thus we can define a branch of the m^{th} root function near $S(0)$, so there exists a holomorphic function R defined in a neighborhood of 0 such that $R(w)^m = S(w)$. Let $\eta(w) = wR(w)$, so

$$T(w) = w^m S(W) = (wR(w))^m = (\eta(w))^m .$$

Then

$$\eta'(w) = wR'(w) + R(w)$$

so $\eta'(0) = R(0) = \sqrt[m]{S(0)} \neq 0$, so near 0 η is invertible by the Implicit Function Theorem. Then $\varphi := \eta \circ \theta$ is also a chart on X defined near P, and since

$$\eta(\theta(P)) = \eta(0) = 0 \cdot R(0) = 0$$

it is also centered at P. Thinking of $z = \eta(w)$ as our new coordinate near P, then we have

$$(\psi \circ F \circ \varphi^{-1})(z) = (\psi \circ F \circ \theta^{-1} \circ \eta^{-1})(z) = T(\eta^{-1}(z)) = (\eta(\eta^{-1}(z)))^m = z^m .$$

□

Lemma 1. *Let $X : f(x, y) = 0$ be a smooth affine plane curve. Consider the projection $\pi : X \to \mathbb{C}$, $(x, y) \mapsto x$. Then π is ramified at $P = (x_0, y_0) \in X$ iff $f_y(P) = 0$.*

Proof. Suppose first that $f_y(P) \neq 0$. Then π is a chart on X near P, so π has multiplicity 1 at P. Conversely, suppose that $f_y(P) = 0$. Then $\rho : (x, y) \mapsto y$ is a chart on X near P. By the Implicit Function Theorem, then there exists a holomorphic function $g(y)$ such that X is locally the graph of g, so $f(g(y), y) = 0$ for all y in the domain of g. Implicitly differentiating with respect to y, we have

$$f_x(g(y), y)g'(y) + f_y(g(y), y) = 0$$

for all y, so in particular

$$0 = f_x(g(y_0), y_0)g'(y_0) + f_y(g(y_0), y_0) = f_x(P)g'(y_0) + f_y(P) = f_x(P)g'(y_0)\,.$$

Since X is smooth and $f_y(P) = 0$, then $f_x(P) \neq 0$, so we must have $g'(y_0) = 0$. $\square$

Example 4. Let $E : Y^2Z = X^3 - Z^3$ and consider the map

$$\begin{aligned} \pi : E &\to \mathbb{P}^1 \\ [X : Y : Z] &\mapsto [X : Z]\,. \end{aligned}$$

On the affine chart U_2 where $Z \neq 0$, E is given by the equation $y^2 = x^3 - 1$ where $x = X/Z$ and $y = Y/Z$ and

$$\pi : [x : y : 1] = [X/Z : Y/Z : 1] \mapsto [X/Z : 1] = [x : 1]\,.$$

Denoting the homogeneous coordinates of $\mathbb{P}^1$ by S, T, then π carries U_2 to the open subset V_1 of $\mathbb{P}^1$ where $T \neq 0$. On V_1 we have the affine coordinate S/T, so the local expression of π as a map $U_2 \to V_1$ is simply $(x, y) \mapsto x$. Letting $f(x) = x^3 - 1$ and

$$h = y^2 - f(x) = y^2 - (x^3 - 1)\,,$$

by the above lemma, π is unramified at all points where $h_y = 2y \neq 0$. Thus it remains to consider the points where $y = 0$, consisting of $(\zeta^j, 0)$ for $j = 0, 1, 2$, where ζ is a primitive third root of unity.

At such a point the projection $(x, y) \mapsto y$ is a coordinate chart, so there exists a holomorphic function $g(w)$ such that

$$0 = h(g(w), w) = w^2 - f(g(w)) = w^2 - (g(w)^3 - 1)$$

and $g(0) = \zeta^j$. Write $g(w) = \sum_{n \geq 0} a_n w^n$, so $a_0 = g(0) = \zeta^j$. Differentiating the above, we find

$$\begin{aligned} 0 &= h_x(g(w), w)g'(w) + h_y(g(w), w) = -3g(w)^2 g'(w) + 2w \\ \implies g'(w) &= \frac{2w}{3g(w)^2} = \frac{2}{3}\frac{w}{g(w)^2}\,. \end{aligned}$$

Thus

$$a_1 = g'(0) = \frac{2}{3}\frac{0}{g(0)^2} = \frac{2}{3}\frac{0}{\zeta^{2j}} = 0$$

so $a_1 = 0$, as expected. Differentiating again, we find

$$g''(w) = \frac{2}{3}\frac{g(w)^2 - w \cdot 2g(w)g'(w)}{g(w)^4}\,.$$

Then

$$2a_2 = g''(0) = \frac{2}{3}\frac{g(0)^2 - 0 \cdot 2g(0)g'(0)}{g(0)^4} = \frac{2}{3}\frac{a_0^2}{a_0{}^4} = \frac{2}{3}\frac{1}{a_0{}^2} = \frac{2}{3}\frac{1}{\zeta^{2j}} = \frac{2}{3}\zeta^j \neq 0\,.$$

Thus $m = 2$ is the smallest $n \geq 1$ such that $a_n \neq 0$, so π has ramification index $e_P(\pi) = 2$ for $P = (\zeta^j, 0)$. (There's one other point we need to check; what is it?)

II.2. **Degree of a morphism.** Given a covering map $\pi : X \to Y$ of connected topological spaces, the fiber $\pi^{-1}(y)$ has the same cardinality for every $y \in Y$. A nonconstant morphism of Riemann surfaces is a covering map except at its ramification points. But if we count these points with multiplicity, i.e., weighted by their ramification indices, then the size of the fiber is again constant.

Proposition 5. *Let $F : X \to Y$ be a nonconstant morphism of compact, connected Riemann surfaces. For each $Q \in Y$, define $d_Q(F)$ to be the sum of the ramification indices of points in the fiber $F^{-1}(Q)$:*

$$d_Q(F) = \sum_{P \in F^{-1}(Q)} e_P(F) .$$

Then $d_Q(F)$ is constant, i.e., independent of Q.

Proof. Idea: show that $Q \mapsto d_Q(F)$ is locally constant. Since Y is connected, then it must be constant. We illustrate the idea of the proof of local constancy with an example. Consider the map

$$\begin{aligned} f : \mathfrak{D} &\to \mathfrak{D} \\ z &\mapsto z^m \end{aligned}$$

for some $m \in \mathbb{Z}_{\geq 1}$. Given $0 \neq w \in \mathfrak{D}$, then w has exactly m preimages (namely the m m^{th} roots of w) and $e_P(f) = 1$ for each of these points P. ($f'(P) \neq 0$ at each of these points.) The point $w = 0$ has only one preimage, namely 0, but has ramification index m. Thus we see that $d_Q(f)$ is constant on $\mathfrak{D}$.

To complete the proof, one uses the Local Normal Form theorem that says that all morphisms locally look like $z \mapsto z^m$ and then keeps a careful count of the number of preimages. □

Definition 6. Let $F : X \to Y$ be a nonconstant morphism of compact, connected Riemann surfaces. The degree of F, denoted $\deg(F)$ is defined to be the integer $d_Q(F)$ for any $Q \in Y$.

Remark 7. So if $d = \deg(F)$, then F is a d-to-1 map.

Corollary 8. *F is an isomorphism iff* $\deg(F) = 1$.

Remark 9. We will later give an algebraic characterization of degree in terms of the function field of a Riemann surface.

Remark 10. For those with a background in algebraic number theory, the constancy of degree may look familiar. Recall the fundamental identity in algebraic number theory: let L/K be an extension of number fields with rings of integers $\mathcal{O}_K$ and $\mathcal{O}_L$. Given a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, then $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$ for some prime ideals $\mathfrak{P}_i$ of $\mathcal{O}_L$ and some $e_i \in \mathbb{Z}_{\geq 1}$. Moreover, we have

$$[L : K] = \sum_{i=1}^{g} e_i f_i$$

where $f_i = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$. We will later make this analogy even sharper using the language of function fields.

III. HYPERELLIPTIC CURVES

Given an affine elliptic curve $y^2 = x^3 + Ax + B$ living inside the affine plane $\mathbb{A}^2$, we can easily find its closure in $\mathbb{P}^2$ simply by homogenizing the defining polynomial. Let's try to generalize this to what are known as hyperelliptic curves. Let $C : y^2 = x^5 - 1$ be an affine plane curve; let's try to determine its closure in $\mathbb{P}^2$ the same way. Is the resulting projective curve smooth?

While there are methods to resolve singularities, a more natural construction is the following. To define hyperelliptic curves, we need a weighted variant of the projective plane, whose definition we sketch below.

Definition 11. Given $g \in \mathbb{Z}_{\geq 1}$ define the weighted projective plane

$$\mathbb{P}(1, g+1, 1) := \frac{\mathbb{C}^3 \setminus \{(0,0,0)\}}{\sim}$$

where $(X, Y, Z) \sim (\lambda X, \lambda^{g+1} Y, \lambda Z)$ for all $\lambda \in \mathbb{C}^\times$.

Remark 12. One can similarly define $\mathbb{P}(a, b, c)$, however there is some strange behavior $\gcd(a, b, c) \neq 1$. Note that $\mathbb{P}(1, 1, 1) = \mathbb{P}^2$.

Just as with the usual projective plane, we have distinguished affine opens U_0, U_1, U_2, where X, Y, and Z are nonzero, respectively. However, the weights come into play when defining the standard open sets. We define

$$U_0 \to \mathbb{A}^2$$
$$[X : Y : Z] = \left[1 : \frac{Y}{X^{g+1}} : \frac{Z}{X}\right] \mapsto \left(\frac{Y}{X^{g+1}}, \frac{Z}{X}\right)$$

$$U_2 \to \mathbb{A}^2$$
$$[X : Y : Z] = \left[\frac{X}{Z} : \frac{Y}{Z^{g+1}} : 1\right] \mapsto \left(\frac{X}{Z}, \frac{Y}{Z^{g+1}}\right).$$

Note the conspicuous absence of a map for U_1! One can define a map on U_1 similarly to the above, but it actually won't be an isomorphism with $\mathbb{A}^2$, but rather the quotient $\mathbb{A}^2/\mu_{g+1}$ of $\mathbb{A}^2$ by the cyclic group of $(g+1)^{\text{st}}$ roots of unity.

However, note that $U_0 \cup U_2$ covers all of $\mathbb{P}(1, g+1, 1)$ except for the single point $[0 : 1 : 0]$ where $X = Z = 0$. It turns out that this point will never lie on our models of hyperelliptic curves, so we can safely ignore it.

Definition 13. A hyperelliptic curve over $\mathbb{C}$ is a smooth plane curve given by an equation of the form

$$Y^2 + h(X, Z)Y = f(X, Z)$$

(called a Weierstrass equation where $f, h \in \mathbb{C}[X, Z]$ are homogeneous of degree $2g + 2$ and $g + 1$, respectively.

Remark 14. Consider $F := Y^2 + h(X, Z)Y - f(X, Z) \in \mathbb{C}[X, Y, Z]$, if we assign X and Z weight 1 and Y weight $g + 1$, then F is weighted homogeneous of degree $2g + 2$.

Since $\mathbb{C}$ has characteristic 0, we can complete the square and obtain a short Weierstrass equation:

$$Y^2 = f(X, Z).$$

Proposition 15. *Let $C : Y^2 = F(X, Z)$ be a hyperelliptic curve, so on the open subset U_2 where $Z \neq 0$, C is given by $y^2 = f(x)$, where $f(x) = F(x, 1)$.*

(a) The map $\iota : (x, y) \mapsto (x, -y)$ extends to an involution (i.e., a morphism such that $\iota^2 = \mathrm{id}$) defined on all of C. (This is called the hyperelliptic involution.*)*

(b) The map

$$\pi : C \to \mathbb{P}^1$$
$$[X : Y : Z] \mapsto [X : Z]$$

is a degree 2 morphism that is ramified above the roots of f, and if f has odd degree, also at the point $[1 : 0 : 0]$.

Proof. We first consider π on U_2, where it is given by $(x, y) \mapsto x$, where $x = X/Z$ and $y = Y/Z$. Given $Q = x_0 \in \mathbb{A}^1$, then $\pi^{-1}(Q)$ consists of the points (x_0, y_0), where y_0 is a solution of the equation

$$y^2 = f(x_0).$$

There are two such solutions, counted with multiplicity, so π has degree 2. By constancy of degree,

$$2 = \deg(\pi) = \sum_{P \in \pi^{-1}(Q)} e_P(\pi)$$

so the ramification values of π are exactly the x_0 such that there is only one solution y_0. This occurs exactly when $f(x_0) = 0$, i.e., x_0 is a root of f.

If f has odd degree, then the weighted homogenization F has a factor of Z. (For instance, if the affine equation for the curve is $y^2 = f(x)$ with $f(x) = x^5 - 1$, then the weighted homogenized equation is $Y^2 = X^5 Z - Z^6$.) Letting $Q = [1 : 0] = \pi([1 : 0 : 0])$, then we compute $\pi^{-1}(Q)$ by subsituting $X = 1$, $Z = 0$ into the equation for C, obtaining $Y^2 = 0$. Thus $\pi^{-1}(Q)$ consists of only one point, hence π is ramified at $[1 : 0 : 0]$. □

IV. Differentials

Some of the notation for defining differentials can be a bit cumbersome, so let's begin with an example to fix ideas.

Example 16. Let's define a differential on $\mathbb{P}^1$. Writing $[X_0 : X_1]$ for the homogeneous coordinates on $\mathbb{P}^1$, recall that we have a holomorphic atlas consisting of the open sets $U_0 = \{X_0 \neq 0\}$ and $U_1 = \{X_1 \neq 0\}$ with coordinate maps

$$\varphi_0 : U_0 \xrightarrow{\sim} \mathbb{C}$$
$$[X_0 : X_1] = [1 : X_1/X_0] \mapsto X_1/X_0$$

$$\varphi_1 : U_1 \xrightarrow{\sim} \mathbb{C}$$
$$[X_0 : X_1] = [X_0/X_1 : 1] \mapsto X_0/X_1.$$

Denote the coordinates on the images of φ_0 and φ_1 by z_0 and z_1, respectively. Consider the differential on dz_1 on $\mathrm{img}(\varphi_1) = \mathbb{C}$. Even if you don't know a rigorous definition for dz_1, you probably know what it is: something we can integrate. (People with background in differential topology will probably say something about covector fields, but it basically amounts to the same thing.) So we have a differential on one chart of $\mathbb{P}^1$: let's see if it extends to all of $\mathbb{P}^1$. Let's work heuristically first. On $U_0 \cap U_1$ we have $z_1 = 1/z_0$, so we should have

$$dz_1 = d(1/z_0) = -\frac{1}{z_0^2} dz_0$$

which gives us the expression for dz_1 or U_0. More rigorously, on $U_0 \cap U_1$ z_1 and z_0 are related by the transition function $\varphi_1 \circ \varphi_0^{-1}$. We have $z_1 = (\varphi_1 \circ \varphi_0^{-1})(z_0)$ which sends

$$z_0 \xmapsto{\varphi_0^{-1}} [1 : z_0] = [1/z_0 : 1] \xmapsto{\varphi_1} 1/z_0$$

so we find

$$dz_1 = (\varphi_1 \circ \varphi_0^{-1})'(z_0)\, dz_0 \,.$$

Definition 17. Given charts $(U_i, \varphi_i), (U_j, \varphi_j)$ on a Riemann surface, and $P \in U_i \cap U_j$ denote the deriviate of their transition function at P by

$$\frac{dz_i}{dz_j}(P) := \left(\varphi_i \circ \varphi_j\right)' \left(\varphi_j(P)\right).$$

Definition 18. A meromorphic differential (one-form) ω on a Riemann surface X consists of an open cover $\{U_i\}_i$ of X and a collection of meromorphic functions $\{f_i : U_i \to \mathbb{C}\}_i$ for each i such that

$$f_j = f_j \frac{dz_i}{dz_j}$$

on $U_i \cap U_j$ for all i, j. If the f_i are holomorphic for all i, then ω is called holomorphic.

Remark 19. We often write this $\omega|_{U_i} = f_i\, dz_i$ and express the compatibility condition by $f_i\, dz_i = f_j\, dz_j$.

Remark 20. For differential geometers, a differential is a section of the cotangent bundle. Our definition is really the same thing. What we've done is specify an invertible sheaf, which is often called a line bundle, by specifying its transition functions.

Definition 21.

- Let X be a Riemann surface with an atlas $\{U_i\}_i$ where the local coordinate on U_i is z_i. Given a meromorphic function $f \in \mathcal{M}(X)$, define

$$\frac{\partial f}{\partial z_i}(P) := (f \circ \varphi_i^{-1})'(\varphi_i(P)) \,.$$

- Given a meromorphic function $f \in \mathcal{M}(X)$, define the meromorphic differential df to be the collection $\left\{ \frac{\partial f}{\partial z_i} \right\}_i$. We often express this by writing $df|_{U_i} = \frac{\partial f}{\partial z_i}\, dz_i$.